□ Review Notes for Linear Algebra – True or False   Last Updated: January 22, 2010

Chapter 2 [Matrix Algebra]

2.1 If $\mathbf{A}$ is a square matrix and $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{A} = \mathbf{I}$ or $\mathbf{A} = -\mathbf{I}$.

False. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

2.2 If $\mathbf{AB} = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.

False. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

2.3 If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are square and $\mathbf{ABC} = \mathbf{O}$, then one of them is $\mathbf{O}$.

False. $\mathbf{A} = \mathbf{B} = \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

2.4 If $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

False. Choose $\mathbf{B} \neq \mathbf{C}$ and $\mathbf{A} = \mathbf{O}$.

2.5 If $\mathbf{A}$ is nonzero and $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

False. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

2.6 The square of a nonzero square matrix must be a nonzero matrix.

False. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2.7 If $\mathbf{AB} = \mathbf{BA}$, then $(\mathbf{A} + \mathbf{B})^3 = \mathbf{A}^3 + 3\mathbf{A}^2\mathbf{B} + 3\mathbf{AB}^2 + \mathbf{B}^3$.

True. $(\mathbf{A}+\mathbf{B})^3 = (\mathbf{A}+\mathbf{B})(\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2) = \mathbf{A}^3 + \mathbf{A}^2\mathbf{B} + \mathbf{ABA} + \mathbf{AB}^2 + \mathbf{BA}^2 + \mathbf{BAB} + \mathbf{B}^2\mathbf{A} + \mathbf{B}^3 = \mathbf{A}^3 + \mathbf{A}^2\mathbf{B} + \mathbf{A}^2\mathbf{B} + \mathbf{AB}^2 + \mathbf{A}^2\mathbf{B} + \mathbf{AB}^2 + \mathbf{AB}^2 + \mathbf{B}^3 = \mathbf{A}^3 + 3\mathbf{A}^2\mathbf{B} + 3\mathbf{AB}^2 + \mathbf{B}^3$.

2.8 An invertible matrix must be a square matrix.

True. $\mathbf{A}$ is invertible $\iff$ $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ for some $\mathbf{B}$. If $\mathbf{A}$ is $m \times n$ and $\mathbf{B}$ is $n \times m$, then we must have $m = n$.

2.9 A non-square matrix can never be invertible.

True. Equivalent to 2.8.

2.10 If $\mathbf{A}$ has a zero row or a zero column, then $\mathbf{A}$ is not invertible.

True. $\mathbf{A}$ has zero determinant and hence not invertible.

2.11 If $\mathbf{A}$ is a square matrix which has no zero rows, then $\mathbf{A}$ is invertible.

False. $\mathbf{A}$ may have a zero column. Then $\mathbf{A}$ is not invertible.

2.12 Let $\mathbf{A}$, $\mathbf{B}$ be invertible matrices of same size. Then $\mathbf{AB}$ is also invertible.

True. $\mathbf{AB}$ is defined and square. $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B}) \neq 0$.

2.13 Let $\mathbf{A}$, $\mathbf{B}$ be invertible matrices of same size. Then $\mathbf{A}+\mathbf{B}$ is also invertible.

False. Let $\mathbf{A}$ invertible. Choose $\mathbf{B}=-\mathbf{A}$ so that $\mathbf{B}$ is invertible. But then $\mathbf{A}+\mathbf{B}=\mathbf{O}$ is not invertible.

2.14 If $\mathbf{AB}$ is equal to the identity matrix, then $\mathbf{A}$ must be an invertible matrix.

False. Choose $\mathbf{A}$, $\mathbf{B}$ non-square such as $\mathbf{A}=\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$.

2.15 $\mathbf{A}$, $\mathbf{B}$ are square matrices. If $\mathbf{AB}=\mathbf{I}$, then $\mathbf{BA}=\mathbf{I}$. Hence, $\mathbf{A}$ is invertible.

True. We prove it by contradiction. In the following proof, we need so-called elementary matrices $\mathbf{E}_i$, for instance, like $\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&-6&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\-4&0&1\end{bmatrix}$. In this case, these three matrices correspond to the elementary row operations $R_2\leftrightarrow R_3$, $-6R_2$, $-4R_1+R_3$, respectively. In general, each row operation always has a corresponding elementary matrix which is also invertible. So, if $\mathbf{A}$ is row equivalent to $\mathbf{B}$ by doing some row operations to $\mathbf{A}$, then we also mean $\mathbf{B}=\mathbf{E}_s\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$ for some invertible $\mathbf{E}_1$, $\mathbf{E}_2$, $\cdots$, $\mathbf{E}_s$.

Proof: Suppose the square matrix $\mathbf{A}$ is not invertible. Then $\mathbf{A}$ is not row equivalent to $\mathbf{I}$ $\Longrightarrow$ the row echelon form of $\mathbf{A}$ must have a zero row $\Longrightarrow$ $\mathbf{E}_s\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{A}$ has a zero row $\Longrightarrow$ $\mathbf{E}_s\cdots\mathbf{E}_2\mathbf{E}_1\mathbf{AB}=\mathbf{E}_s\cdots\mathbf{E}_2\mathbf{E}_1=$ invertible matrix also has a zero row $\Longrightarrow$ a contradiction. This proves $\mathbf{A}$ must be invertible and hence $\mathbf{BA}=\mathbf{I}$.

2.16 For square matrix $\mathbf{A}$, $\mathbf{AA}^t=\mathbf{I}$ if and only if $\mathbf{A}^t\mathbf{A}=\mathbf{I}$.

True. If $\mathbf{A}$, $\mathbf{B}$ are square matrices, by 2.15, then $\mathbf{AB}=\mathbf{I}\iff\mathbf{BA}=\mathbf{I}$.

2.17 If $\mathbf{AB}$ is invertible, then $\mathbf{BA}$ is invertible.

False. Take $\mathbf{A}$, $\mathbf{B}$ in 2.14. Then $\mathbf{AB}=\mathbf{I}$ but $\mathbf{BA}$ has a zero row and hence not invertible.

2.18 If $\mathbf{A}^2\neq\mathbf{O}$, then $\mathbf{A}$ is invertible.

False. $\mathbf{A}=\begin{bmatrix}2&4\\1&2\end{bmatrix}$. $\mathbf{A}^2\neq\mathbf{O}$ but $\mathbf{A}$ not invertible ($\det\mathbf{A}=0$).

2.19 If $\mathbf{A}$ is invertible, then $\mathbf{A}^2\neq\mathbf{O}$.

True. Prove by contradiction. Suppose $\mathbf{A}^2=\mathbf{O}$. Then $\det\mathbf{A}^2=\det\mathbf{O}=0$. Because $\det\mathbf{A}^2=(\det\mathbf{A})^2$ and hence $\det\mathbf{A}=0$ and $\mathbf{A}$ is not invertible.

2.20 If $\mathbf{A}$ is a square matrix and $\mathbf{A}^2+7\mathbf{A}-\mathbf{I}=\mathbf{O}$, then $\mathbf{A}$ is invertible.

True. $\mathbf{A}$ is square and $\mathbf{A}(\mathbf{A}+7\mathbf{I})=\mathbf{I}\implies\mathbf{A}^{-1}=\mathbf{A}+7\mathbf{I}$, by 2.15.

2.21 A symmetric matrix must be a square matrix.

True. If $\mathbf{A}$ is an $m\times n$ matrix, then $\mathbf{A}^t$ is $n\times m$. If, furthermore, $\mathbf{A}$ is symmetric, then by definition, $\mathbf{A}^t=\mathbf{A}$ which implies that $m=n$. Otherwise, they won't have the same size.

2.22 If $\mathbf{A}$ is symmetric, so are $\mathbf{A}^{-1}$ (if exists) and $\mathbf{A}^3$.

True. $\mathbf{A}$ symmetric $\iff\mathbf{A}^t=\mathbf{A}$. Then $(\mathbf{A}^{-1})^t=(\mathbf{A}^t)^{-1}=\mathbf{A}^{-1}$ and $(\mathbf{A}^3)^t=(\mathbf{A}^t)^3=\mathbf{A}^3$.

2.23 If $\mathbf{B}=\mathbf{A}^t\mathbf{A}$, then $2\mathbf{B}$ is symmetric.

True. $(2\mathbf{B})^t=2\mathbf{B}^t=2(\mathbf{A}^t\mathbf{A})^t=2\mathbf{A}^t(\mathbf{A}^t)^t=2\mathbf{A}^t\mathbf{A}=2\mathbf{B}$.

2.24 If $\mathbf{A}$ is symmetric, so is $f(\mathbf{A})$, for any polynomial $f(x)$.

True. By definition, $\mathbf{A}$ symmetric $\Longleftrightarrow \mathbf{A}^t = \mathbf{A}$. In general we also have $(\mathbf{A}+\mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t$ and $(c\mathbf{A})^t = c\mathbf{A}^t$. Hence, if $\mathbf{A}$ is symmetric, then for any polynomial $f(x)$, the transpose of $f(\mathbf{A})$ must equal to $f(\mathbf{A})$ itself. Hence, $f(\mathbf{A})$ is symmetric.

2.25 $\det(\mathbf{A}+\mathbf{B}) = \det\mathbf{A} + \det\mathbf{B}$.

False. $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2.26 $\det(k\mathbf{A}) = k \cdot \det\mathbf{A}$, for any integer k.

False. $\mathbf{A} = \mathbf{I}$. Then $\det\mathbf{A} = 1$, $\det(2\mathbf{A}) = \det\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 \neq 2 \cdot \det\mathbf{A}$.

2.27 $\det\mathbf{A}^t = (-1)\det\mathbf{A}$.

False. $\mathbf{A} = \mathbf{I}$. Then $\mathbf{A}^t = \mathbf{A}$.

2.28 Three elementary row operations do not change the determinant of a square matrix.

False. (1) If $\mathbf{B}$ is obtained from $\mathbf{A}$ by interchanging any two rows of $\mathbf{A}$, then $\det\mathbf{B} = -\det\mathbf{A}$. (2) If $\mathbf{B}$ is obtained from $\mathbf{A}$ by multiplying a row of $\mathbf{A}$ by k, then $\det\mathbf{B} = k \cdot \det\mathbf{A}$.

2.29 A row replacement operation does not change the determinant of a matrix.

True. Let $\mathbf{A}$ be a square matrix. (1) If a multiple of one row of $\mathbf{A}$ is added to another row to produce a matrix $\mathbf{B}$, then $\det\mathbf{B} = \det\mathbf{A}$. (2) If two rows of $\mathbf{A}$ are interchanged to produce $\mathbf{B}$, then $\det\mathbf{B} = -\det\mathbf{A}$. (3) If one row of $\mathbf{A}$ is multiplied by k to produce $\mathbf{B}$, then $\det\mathbf{B} = k \cdot \det\mathbf{A}$.

2.30 If $\mathbf{A}$ is row equivalent to $\mathbf{B}$, then $\det\mathbf{A} = \det\mathbf{B}$.

False. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $\mathbf{A} \to \mathbf{I}$, and $\det\mathbf{A} = -2 \neq 1$.

2.31 If $\mathbf{A}$ is row equivalent to $\mathbf{B}$, then $\det\mathbf{A}$ and $\det\mathbf{B}$ are either both zero or both nonzero.

True. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by either one of the three elementary row operations, then $\det\mathbf{B} = k \cdot \det\mathbf{A}$, where $k \neq 0$.

2.32 The determinant of a triangular matrix is the sum of the entries on the main diagonal.

False. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $\det\mathbf{A} = 1$. The correct statement should be: The determinant of a (lower or upper) triangular matrix is the *product* of the entries on the diagonal.

2.33 Let $\mathbf{A}$ be a square matrix without zero rows and columns. Then $\mathbf{A}$ must be row equivalent to the identity matrix of same size.

False. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

2.34 If $\mathbf{A}$, $\mathbf{B}$ are nonzero square matrices and $\mathbf{A}$ is row equivalent to $\mathbf{B}$, then both $\mathbf{A}$, $\mathbf{B}$ are invertible.

False. $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

2.35 If $\mathbf{A}$ is an invertible matrix and $\mathbf{B}$ is row equivalent to $\mathbf{A}$, then $\mathbf{B}$ is also invertible.

True. 2.31 shows that, if $\mathbf{A}$ is row equivalent to $\mathbf{B}$, then their determinants are both zero or both nonzero. Thus, $\mathbf{A}$ is invertible $\Longrightarrow \det\mathbf{A} \neq 0 \Longrightarrow \det\mathbf{B} \neq 0 \Longrightarrow \mathbf{B}$ is invertible.