Chapter 6 Mathematical Expectation / Expected Value(期望值)

One of the most important concepts in probability theory is that of the expectation of a random variable. The expectation of X is called the mean and is usually denoted by μ; it indicates the “center” of the probability distribution in the sense of a center of gravity. The expected value of $(X - \mu)^2$ is called the invariance, and is denoted by σ^2. Its positive square root σ is called the standard deviation. For most common distributions, the interval $(\mu - 3\sigma, \mu + 3\sigma)$ contains almost all of the probability.

Definition 6.1: If X is a *discrete* random variable having a probability density function $f(x)$, the expectation or the expected value of X, denoted by E(X), is defined by

$$E(X) = \sum_{x \in I} x f(x), \qquad \text{where} \quad f(x) = P(X = x), \quad I = \text{Image}(X).$$

In words, the expected valued of X is a weighted average (or mean) of the possible values that X can take on, each value being weighted by the probability that X assumes it. For example, if the probability density function of X is given by

$$f(0) = f(1) = 1/2,$$

then $E(X) = 0\,(1/2) + 1\,(1/2) = 1/2$ is just the ordinary average of the two possible values 0 and 1 that X can assume. On the other hand, if

$$f(0) = 1/3, \qquad f(1) = 2/3,$$

then $E(X) = 0\,(1/3) + 1\,(2/3) = 2/3$ is a weighted average of the two possible values 0 and 1, where the value 1 is given twice as much weight as the value 0, since $f(1) = 2f(0)$.

Exercise 6.2: Find E(X) where X is the outcome when we roll a fair die.

Exercise 6.3: Let X be the maximum numbers on the dice when 3 fair dice are rolled. Evaluate the expectation E(X).

Exercise 6.4： An urn contains 3 white, 4 blue, 6 red and 4 black balls. Four balls are randomly selected by a player from the urn. The player will win 1 dollar if each white ball is selected; win 50cents if each blue ball is selected; and lose 1.5 dollars if each red ball is selected. Let X be the amount of money of the player.

(i) What is the image of X, i.e., I = Image(X) ?

(ii) For each of the element a in I, find $P(X = a)$.

(iii) Find the expectation E(X).

Example 6.5： If X is a binomial random variable with parameters *n* and *p*, compute its expectation.

Proof：

$$E(X) = \sum_{i=0}^{n} i\ C_i^n p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} i \frac{n!}{(n-i)!\,i!} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!\,(i-1)!} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} C_{i-1}^{n-1} p^{i-1} (1-p)^{n-i}$$

$$= np\ [p + (1-p)]^{n-1} = np$$ 。

Definition 6.6： If X is a *continuous* random variable having a probability density function $f(x)$, the expectation or the expected value of X, denoted by E(X), is defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx .$$

Exercise 6.7： Let X be uniformly distributed over the interval (a, b). Find the expectation of X.

Example 6.8： Find the expectation of X when X is a normal random variable with parameters μ and σ.

Proof：

$$E(X) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} (x-\mu) e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \mu e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{-\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\left(e^{\frac{-(x-\mu)^2}{2\sigma^2}} \right) + \frac{\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2}\,dz \qquad \left(z = \frac{x-\mu}{\sqrt{2}\sigma} \right)$$

$$= \frac{-\sigma}{\sqrt{2\pi}}\left[e^{-\infty} - e^{-\infty}\right] + \frac{\mu}{\sqrt{\pi}}\left[\sqrt{\pi}\right] = \mu .$$

Definition 6.9： If X is a random variable with expected value μ, then the variance of X is defined (for any type of random variable) by

$$\mathrm{Var}(X) = E((X-\mu)^2).$$

Standard deviation is the positive square root of the variance.

Remark. $\mathrm{Var}(X) = E(X^2) - (E(X))^2$.

Example 6.10： Find the variance of X if X ~ Bin(n, p).

Proof：

$$E(X^2) = \sum_{i=0}^{n} i^2 C_i^n p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} i(i-1)\frac{n!}{(n-i)!i!} p^i (1-p)^{n-i} + \sum_{i=0}^{n} iC_i^n p^i (1-p)^{n-i}$$

$$= \sum_{i=2}^{n} \frac{n!}{(n-i)!(i-2)!} p^i (1-p)^{n-i} + E(X)$$

$$= n(n-1)p^2 \sum_{i=2}^{n} C_{i-2}^{n-2} p^{i-2}(1-p)^{n-i} + np$$

$$= n(n-1)p^2 \left[p + (1-p)\right]^{n-2} + np$$

$$= n(n-1)p^2 + np,$$

$$Var(X) = E((X-\mu)^2)$$

$$= E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + E(\mu^2)$$

$$= E(X^2) - \mu^2$$

$$= n(n-1)p^2 + np - n^2p^2 = np(1-p).$$

Example 6.11： Find the variance of X if X ~ N(μ, σ).

Proof：

$$Var(X) = E((X-\mu)^2)$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2 e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2 e^{\frac{-y^2}{2}} dy \qquad \left(y = \frac{x-\mu}{\sigma}\right)$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left(-ye^{\frac{-y^2}{2}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy\right)$$

$$= \sigma^2\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-y^2}{2}} dy\right) = \sigma^2 .$$

Exercise 6.12： Repeat **Exercise 6.4** and find the variance of X, Var(X).

Exercise 6.13： An urn contains 3 white, 3 red and 5 black balls. Three balls are randomly selected by a player from the urn. The player will win 1 dollar if each white ball is selected and will lose 1 dollar if each red ball is selected. Let X denote the amount of money that the player has earned. Find (i) E(X), (ii) Var(X).